TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
SOLUTION SKETCHES TO HOMEWORK 11

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Homework 11.1 (Cauchy–Riemann equations and $\mathbf{C}$-linearity on $\mathbf{C}^n$)**.** Identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ and let $U \subset \mathbf{C}^n$. Consider a differentiable function $f\colon U \to \mathbf{C}$. Then at each point $a \in U$ there exists an $\mathbf{R}$-linear mapping $\mathrm{D}f(a)\colon \mathbf{R}^{2n} \to \mathbf{C}$ such that

$$\lim_{\substack{h\to 0,\\ h\neq 0}} \frac{|f(a+h) - f(a) - Df(a)h|}{|h|} = 0.$$

Show $\mathrm{D}f(a)$ is $\mathbf{C}$-linear if and only if

$$\frac{\partial}{\partial \overline{z}_j} f(a) = 0$$

for every $j \in \{1,\dots,n\}$, where $2\partial/\partial\overline{z}_j := \partial/\partial x_j + \mathrm{i}\,\partial/\partial y_j$ and $z = x + \mathrm{i}y$ with $x, y \in \mathbf{R}^n$.

Solution. We identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ via $x + \mathrm{i}y := (x, y)$. The $\mathbf{C}$-linearity of $\mathrm{D}f(a)$ means that for all values $\lambda_1 + \mathrm{i}\lambda_2$ with $\lambda_1, \lambda_2 \in \mathbf{R}$ and all $x, y \in \mathbf{R}^n$ we have

$$\mathrm{D}f(a)(\lambda_1 + \mathrm{i}\lambda_2)(x + \mathrm{i}y) = (\lambda_1 + \mathrm{i}\lambda_2)\,\mathrm{D}f(a)(x + \mathrm{i}y).$$

Since $\mathrm{D}f(a)$ is $\mathbf{R}$-linear, this is equivalent to showing

$$\mathrm{D}f(a)(-y, x) = \mathrm{i}\,\mathrm{D}f(a)(x, y)$$

for all $x, y \in \mathbf{R}^n$ or, considering real and imaginary parts,

$$\begin{aligned}\Re\mathrm{D}f(a)(-y, x) &= -\Im\mathrm{D}f(a)(x, y),\\ \Im\mathrm{D}f(a)(-y, x) &= \Re\mathrm{D}f(a)(x, y).\end{aligned}$$

It is enough to check these equations along basis vectors x and y. This leads to

$$\begin{aligned}\Re\Big[-\frac{\partial f}{\partial x_j}(a)\Big] &= -\Im\frac{\partial f}{\partial y_j}(a),\\ \Re\frac{\partial f}{\partial y_j}(a) &= -\Im\frac{\partial f}{\partial x_j}(a),\\ \Im\Big[-\frac{\partial f}{\partial x_j}(a)\Big] &= \Re\frac{\partial f}{\partial y_j}(a),\\ \Im\frac{\partial f}{\partial y_j}(a) &= \Re\frac{\partial f}{\partial x_j}(a)\end{aligned}$$

for every $j \in \{1,\dots,n\}$. Note the first and fourth as well as the second and third equation are equivalent. Moreover the first and second can be reformulated by

$$\frac{\partial f}{\partial x_j}(a) = \Im\frac{\partial f}{\partial y_j}(a) - \mathrm{i}\,\Re\frac{\partial f}{\partial y_j}(a) = -\mathrm{i}\,\frac{\partial f}{\partial y_j}(a).$$

This is clearly equivalent to $\partial f/\partial\overline{z}_j(a) = 0$.

Homework 11.2 (Slicing method in action)**.** In this exercise we transfer some well-known results from one-dimensional complex analysis to the several variables setting. Show the following statements.

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a. **Liouville's theorem**. Every bounded entire function $f\colon \mathbf{C}^n \to \mathbf{C}$ is constant.
b. **Identity theorem**. Let $D \subset \mathbf{C}^n$ be a domain and $f\colon D \to \mathbf{C}$ be holomorphic. If f vanishes identically on $B_r(a)$ for some $a \in D$ and $r > 0$, then $f = 0$.
c. **Open mapping theorem**. Let $D \subset \mathbf{C}^n$ be a domain and $f\colon D \to \mathbf{C}$ be nonconstant and holomorphic. Then $f(D)$ is again a domain.
d. **Maximum principle**. Let $D \subset \mathbf{C}^n$ be a domain and $f\colon D \to \mathbf{C}$ be holomorphic. If $|f|$ attains its maximum on D then f is constant.

Solution. a. Let $a, b \in \mathbf{C}^n$. With the notation of Lemma 8.3, consider the slicing $f_{a,b-a}$ which satisfies $f_{a,b-a}(0) = f(a)$ and $f_{a,b-a}(1) = f(b)$. Note $f_{a,b-a}$ is holomorphic on $\mathbf{C}$ and bounded, so that the standard Liouville theorem implies $f_{a,b-a}(0) = f_{a,b-a}(1)$, which means that $f(a) = f(b)$. Since $b \in \mathbf{C}^n$ was arbitrary we conclude that f is constant.

b. Define $U := \{z \in D : f = 0 \text{ in a neighborhood of } z\}$. Then $a \in U$. Moreover, U is open. We claim that U is also closed in D. Then by connectedness of D it follows $U = D$.

Let $(z_n)_{n\in\mathbf{N}}$ be a sequence in U converging to $z \in D$. Assume $z \notin U$. Then for each $r > 0$ there exists $z_r \in B_r(z) \cap D$ such that $f(z_r) \neq 0$. Let z_n be such that $z_n \in B_r(z)$ for some $r > 0$ such that $B_{2r}(z) \subset D$. Note that $z_r \neq z_n$ since $z_r \notin U$. Then we consider the slicing $f_{z_n,z_r-z_n}\colon D_{z_n,z_r-z_n} \to \mathbf{C}$. By convexity of $B_r(z)$ we have $[0,1] \subset D_{z_n,z_r-z_n}$. Since D_{z_n,z_r-z_n} is open, we further find a cylinder of the form $Z_\delta := (-\delta, 1+\delta) + \mathrm{i}(-\delta,\delta)$ such that $Z_\delta \subset D_{z_n,z_r-z_n}$. Moreover, since $z_n \in U$ it follows that there exists $\tau_n > 0$ such that $f_{z_n,z_r-z_n} = 0$ on $[0,\tau_n]$. Hence by the one-dimensional identity theorem it follows that f_{z_n,z_r-z_n} vanishes on Z_δ. This implies the contradiction $f(z_r) = f_{z_n,z_r-z_n}(1) = 0$.

c. By continuity, $f(D)$ is path-connected. Thus it only remains to show $f(D)$ is open. Let $w \in f(D)$ and consider $z \in D$ such that $f(z) = w$. Consider a ball $B_r(z) \subset D$. Then the restriction $f\big|_{B_r(z)}$ cannot be constant by the identity theorem, so that there exists a point $p \in B_r(z)$ such that $f(p) \neq f(z)$. In particular, the function $f_{p,z-p}\colon D_{p,z-p} \to \mathbf{C}$ is not constant, so that by the one-dimensional open mapping theorem the set $f_{p,z-p}(D_{p,z-p})$ contains an open neighborhood of $f(z)$. (More precisely, we consider $f_{p,z-p}$ restricted to a suitable cylinder as in b. to have a connected subset where it is not constant.) Since $w \in f_{p,z-p}(D_{p,z-p}) \subset f(D)$ we deduce the claim.

d. If $|f|$ attains its maximum on D in a point a and f is not constant, then there cannot exist a neighborhood of $f(a)$ in $f(D)$ since $|f(z)| \le |f(a)|$ for all $z \in D$. This contradicts the open mapping theorem.

Homework 11.3 (Failure of the open mapping theorem in the fully vectorial case)**.** In Homework 11.2 we proved the open mapping theorem for functions with target domain $\mathbf{C}$. Here we show that it is false for vectorial functions $f\colon D \to \mathbf{C}^m$, where $m \ge 2$, even when no component is constant. Define $f\colon \mathbf{C}^2 \to \mathbf{C}^2$ by $f(z_1,z_2) := (z_1, z_1 z_2)$. Show f is holomorphic yet not an open map[1].

Solution. The function f is holomorphic since each component is a polynomial. Note $(0,0) \in f(B_r(0))$ for any $r > 0$. Fix such an r. We claim there exists a sequence $(w_n)_{n\in\mathbf{N}}$ in $\mathbf{C}^2$ such that $w_n \to (0,0)$ in $\mathbf{C}^2$ as $n \to \infty$ and $w_n \notin f(B_r(0))$ for all $n \in \mathbf{N}$. Indeed, define $w_n = (1/n^2, 1/n)$. Then $f(z) = w_n$ if and only if $z_1 = 1/n^2$ and $z_2 = n$. But then

$$\liminf_{n\to\infty} |(z_1,z_2)| \ge \liminf_{n\to\infty} \left[n|(0,1)| - \frac{1}{n^2}|(1,0)| \right] = \infty.$$

This means $(z_1,z_2) \notin B_r(0)$ for sufficiently large $n \in \mathbf{N}$. Hence $f(B_r(0))$ is not open.

[1]**Hint.** In order to guess where the map is not open one can look where its differential is not invertible.

Homework 11.4 (Power series in several variables*). a. For each series below, determine for each series below the largest open set $U \subset \mathbf{C}^2$ where it converges absolutely. Is it convex?

- $\sum_{n=0}^{\infty} z^n w^n$.
- $\sum_{n=1}^{\infty} z^n w^{n!}$.

b. Let $F(z) := \sum_{\alpha \in \mathbf{N}_0^n} c_\alpha z^\alpha$ be a formal power series centered at the origin. Show that if $z \in \mathbf{C}^n$ is such that $F(z)$ converges absolutely, then $F(\lambda_1 z_1, \ldots, \lambda_n z_n)$ also converges absolutely provided $|\lambda_i| \leq 1$ for every $i \in \{1, \ldots, n\}$.

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